JOINT EFFECT OF SLIP AND PERMEABILITY CONDITIONS OF SURFACE ON FLOW AND HEAT EXCHANGE IN THE VICINITY OF A CYLINDER CRITICAL POINT

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An investigation was made of the boundary layer of a weakly rarefied gas close to a cylinder critical point with suction and exhaust.

Let us consider a laminar boundary layer of a homogeneous rarefied gas close to a critical point of a permeable cylinder.

The boundary-layer equations are as follows:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0,$$
$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),$$
$$\frac{\partial p}{\partial y}=0, \tag{1}$$
$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=u\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Pr}}\cdot\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2,$$
$$p=\frac{\gamma-1}{\gamma}\rho h,\quad \frac{\mu}{\mu_0}=F\left(\frac{h}{h_0}\right),$$

where the x and y axes are directed along the surface and along a normal to it respectively.

The slip and temperature jump conditions are valid on the surface of the cylinder. If the rarefication effect is ignored the hydrodynamic problem is self-simulating [2] in the case of constant mass flow-rate of gas on the wall. It is assumed that this is the case here. Then the boundary conditions on the surface can be written as follows:

$$u=\frac{2-\sigma_\tau}{\sigma_\tau}\sqrt{\gamma\frac{\pi}{2}}\,\frac{\mu}{\rho\sqrt{(\gamma-1)h}}\cdot\frac{\partial u}{\partial y},$$
$$h=h_w+\frac{2-\sigma_E}{\sigma_E}\cdot\frac{\gamma}{2\mathrm{Pr}(\gamma-1)}\sqrt{\gamma\frac{\pi}{2}}\cdot\frac{\mu}{\rho\sqrt{(\gamma-1)h}}\cdot\frac{\partial h}{\partial y} \tag{2}$$
$$-\frac{2-\sigma_E}{4\sigma_E}\sqrt{\frac{\pi\gamma h}{2(\gamma-1)}}\,v,\quad \rho v=\mathrm{const}.$$

Away from the surface the values of u and h should approach their corresponding values for the free-stream flow:

for $y\to\infty$

$$u=U_\infty,\quad h=h_\infty. \tag{3}$$

The velocity distribution in potential flow close to a cylinder critical point in the case of two dimensional flow-past is given by the formula

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$$U_\infty = Ax,$$

where A > 0. As a result of U_∞ changing with the x coordinate and the term $u(\partial p/\partial x)$ remaining unchanged in the energy equation, the enthalpy h_∞ is given by the expression

$$h_\infty = h_0 - \frac{A^2x^2}{2}.$$

Since the dynamic viscosity coefficient μ is a function of the absolute temperature T (or of enthalpy h) the equation of motion cannot be solved independently of the energy equation. However, by using the linear relation between μ and h [3], in accordance with which

$$\frac{\mu}{\mu_0} = C\frac{h}{h_0}, \quad C = \left(\frac{T_w}{T_0}\right)^{1/2}\frac{T_0 + T_S}{T_w + T_S},$$

where T_S is the Sutherland constant, the equation can be made autonomous.

The Dorodnitsyn change of variables:

$$\xi = x, \quad \eta = \int_0^y \frac{\rho}{\rho_0}\,dy$$

is now introduced and also instead of v the function

$$v^* = \frac{\rho}{\rho_0}v + u\frac{\partial\eta}{\partial x}.$$

The following notation is used:

$$k = \frac{2-\sigma_\tau}{\sigma_\tau}\sqrt{\gamma\frac{\pi}{2}}\cdot\frac{C\mu_0}{\rho_0\sqrt{(\gamma-1)h_0}},$$

$$\alpha = \frac{(2-\sigma_E)\,\sigma_\tau}{(2-\sigma_\tau)\,\sigma_E}\cdot\frac{\gamma}{2\Pr(\gamma-1)},$$

$$\beta_1 = -\frac{2-\sigma_E}{4\sigma_E}\cdot\sqrt{\frac{\pi\gamma}{2(\gamma-1)}}\cdot\frac{h_w^{3/2}}{h_0}.$$

Equations (1) and the boundary conditions (2) and (3) are made dimensionless by introducing the variables $\lambda = \eta\sqrt{A/C\nu_0}$, $\vartheta = A\xi/\sqrt{c_pT_0}$, $\bar{h} = h/h_0$, and the flow function

$$\psi = \sqrt{C\nu_0A}\,\xi\varphi(\lambda). \tag{4}$$

Omitting the bar over the dimensionless enthalpy one obtains (primes denote differentiation with respect to λ):

$$\left.\begin{aligned} &\varphi''' + \varphi\varphi'' = \varphi'^2 - 1, \\ &\frac{\partial^2 h}{\partial\lambda^2} - \Pr\vartheta\varphi'\frac{\partial h}{d\vartheta} + \Pr\varphi\frac{\partial h}{\partial\lambda} = -\Pr\vartheta^2\varphi''^2 + \Pr\vartheta^2\varphi'; \end{aligned}\right\} \tag{5}$$

for $\lambda = 0$

$$\varphi = B, \quad \varphi' = k_1\varphi'', \quad h = h_w + \alpha k_1\frac{\partial h}{\partial\lambda} + \beta, \tag{6}$$

for $\lambda \to \infty$

$$\varphi' = 1, \quad h = 1 - \frac{\vartheta^2}{2},$$

where

$$B = -\frac{v_0^*}{\sqrt{C\nu_0A}}, \quad k_1 = k\sqrt{\frac{Ah_w}{C\nu_0}}, \quad \beta = \frac{\beta_1 v_0^*}{c_pT_0}.$$

Thus in spite of slipping the equation of motion remains self-simulating. This is only characteristic for a flow in the vicinity of a cylindrical point since in this case the coordinate λ is independent of ξ; however, for a boundary layer on a semi-infinite plate there is no self-simulation if, for example, the effects of rarefication are taken into account [4].

The parameter k_1 with an accuracy up to a multiplier is equal to the ratio of the average length of free path l_0 to the boundary layer thickness δ, which is constant close to a critical point. It is advisable, therefore, when solving the first equation of (5) that a series expansion be used with respect to the constant small parameter, namely:

$$\varphi(\lambda) = \varphi_1(\lambda) + k_1\varphi_2(\lambda) + \ldots \tag{7}$$

Only the first two terms are retained in the expansion (7), since in the boundary layer equations only terms of the order of smallness of l_0/δ are kept.

By substituting (7) in the first equation of (5) and the corresponding boundary conditions in (6) the following equations and boundary conditions are obtained for the first and second approximation:

$$\left.\begin{aligned} &\varphi_1''' + \varphi_1\varphi_1'' = \varphi_1'^2 - 1, \\ &\varphi_1(0) = B, \ \varphi_1'(0) = 0, \\ &\varphi_1'(\infty) = 1; \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} &\varphi_2''' + \varphi_1\varphi_2'' - 2\varphi_1'\varphi_2' + \varphi_1''\varphi_2 = 0, \\ &\varphi_2(0) = 0, \ \varphi_2'(0) = \varphi_1''(0), \\ &\varphi_2'(\infty) = 0. \end{aligned}\right\} \tag{9}$$

The relation (8) is a Folkner–Scan equation for a permeable surface ($m = 1$). The results of numerical integration of that equation for various values of B can be found in [5].

A solution of Eq. (9) satisfying the boundary conditions is given by

$$\varphi_2 = \varphi_1'. \tag{10}$$

The friction on the cylinder surface is now found:

$$\tau_w = \rho_0 U_\infty^2 \sqrt{\frac{C}{\mathrm{Re}_0}}\, \varphi_1''(0) \left[1 - k_1 \frac{B\varphi_1''(0) + 1}{\varphi_1''(0)}\right], \tag{11}$$

where $\mathrm{Re}_0 = U_\infty x/\nu_0$. The first of the components, containing k_1, is due to the joint effect of slip and surface permeability, the second component is due to slip only, i.e., in this case the friction depends on the slip also in the case of an impermeable cylinder in contrast to the surface friction on the plate [4].

It follows from (11) that for $k_1 \neq 0$ for any B the surface friction is smaller than for $k_1 = 0$, since $\varphi_1''(0) > 0$ and $B\varphi_1''(0) + 1 = -\varphi_1'''(0) > 0$; however, in the case of exhaust both components, containing k_1, are of the same sign; for suction, however, they have opposite signs. Therefore, the surface friction is considerably more reduced by the slip in the case of exhaust than in the case of suction (see Table 1).

The solution of the second equation of the system (5), which is the energy equation, is sought in the form of a sum,

$$h(\vartheta, \lambda) = \chi_1(\lambda) + \vartheta^2\chi_2(\lambda). \tag{12}$$

By substituting the sum (12) into the energy equation and the corresponding boundary conditions in (6), the following equation and boundary conditions are obtained for the functions $\chi_1(\lambda)$ and $\chi_2(\lambda)$:

$$\left.\begin{aligned} &\chi_1'' + \Pr\varphi\chi_1' = 0, \\ &\chi_1(0) + a\chi_1'(0) = b, \\ &\chi_1(\infty) = 1; \end{aligned}\right\}$$

$$\left.\begin{aligned} &\chi_2'' + \Pr\varphi\chi_2' - 2\Pr\varphi'\chi_2 = -\Pr\varphi''^2 + \Pr\varphi', \\ &\chi_2(0) + a\chi_2'(0) = 0, \\ &\chi_2(\infty) = -\frac{1}{2}, \end{aligned}\right\}$$

where $a = -\alpha k_1$, $b = h_W + \beta$.

A solution of Eq. (13) can be written as follows:

$$\chi_1 = C_1 + C_2 \int_0^\lambda \exp\left\{-\Pr\int_0^{\lambda_1} \varphi\, d\lambda_1\right\} d\lambda.$$

TABLE 1. Values of Surface Friction and Heat Flow Towards Cylinder Surface as Dependent on Rarefication Rate and Transverse Velocity

B	k_1	$\frac{\tau_w}{\frac{1}{2}\rho_0 U_\infty^2}\sqrt{\frac{\mathrm{Re}_0}{C}}$	$\frac{q_w}{\rho_0 U_\infty c_p T_0}\mathrm{Pr}\sqrt{\frac{\mathrm{Re}_0}{C}}$
1,9265	0	5,216	0,4672
	0,1	4,011	0,3898
0	0	2,465	0,1576
	0,1	2,265	0,1520
−1,198	0	1,373	0,0264
	0,1	1,337	0,0299

The constants C_1 and C_2 are determined by using the boundary conditions (13):

$$C_1 = 1 - C_2 I_\infty, \quad C_2 = \frac{1-b}{I_\infty - a},$$

where

$$I_\infty = \int_0^\infty \exp\left\{-\mathrm{Pr}\int_0^{\lambda_1} \varphi d\lambda_1\right\} d\lambda.$$

It should be noted before solving Eq. (14) that for large λ ($\lambda > \lambda_0$, where λ_0 can be found from the tables in [5]) the function φ_1 can be represented with a large degree of accuracy as

$$\varphi_1 = \lambda + E$$

(the constant E is also evaluated from the tables); then from (7) one obtains

$$\varphi = \lambda + E + k_1.$$

By introducing the change of variable

$$z = \sqrt{\mathrm{Pr}}\,(\lambda + E + k_1), \quad t = iz/\sqrt{2},$$

the following asymptotic equation is obtained from (14):

$$\frac{d^2\chi_2}{dt^2} - 2t\frac{d\chi_2}{dt} + 4\chi_2 = -2.$$

Its solution, which satisfies the condition $\chi_2(\infty) = -1/2$, is given by [3]

$$\chi_2 = N(z^2+1)I(z) - \frac{1}{2},$$

where N is an undetermined constant and

$$I(z) = \int_z^\infty \frac{\exp\{-z^2\}}{(z^2+1)^2} dz.$$

By eliminating N from χ_2 and χ_2', one finds the boundary condition for some λ_1 ($\lambda_1 > \lambda_0$):

$$\sqrt{\mathrm{Pr}}\left[4z_1 I(z_1) - \frac{\exp\{-z_1^2\}}{2(z_1^2+1)}\right]\left[\chi_2(\lambda_1) + \frac{1}{2}\right] = 2(z_1^2+1)I(z_1)\chi_2'(\lambda_1).$$

Equation (14) together with the boundary condition for $\lambda = 0$ and the condition (15) was solved by the trial method on the digital computer Minsk-22.

The actual heat flow to the cylinder surface, the heat from the friction forces being taken into account, is as follows:

$$q_w = \frac{\rho_0 U_\infty c_p T_0}{\mathrm{Pr}}\sqrt{\frac{C}{\mathrm{Re}_0}}\left[C_2 + \vartheta^2\chi_2'(0) + \mathrm{Pr}\,k_1\vartheta^2\varphi_1''(0)\right].$$

In Table 1 the values of the quantities $\frac{\tau_w}{\frac{1}{2}\rho_0 U_\infty^2}\sqrt{\frac{\mathrm{Re}_0}{C}}$ and $\frac{q_w}{\rho_0 U_\infty c_p T_0}\mathrm{Pr}\sqrt{\frac{\mathrm{Re}_0}{C}}$ for the air are given for $\sigma_T = \sigma_E = 1$, $T_w = 400°K$, $T_0 = 600°K$; they correspond to two values of B found in [5] (the heat flow is computed for $\vartheta = 0.5$). It can be seen from Table 1 that the inclusion of the slip conditions in suction and exhaust leads to a greater change in the heat flow than in the case of impermeable surface.

In conclusion it should be mentioned that in the case of exhaust the appearance of slip and of temperature jump have a much greater effect on the heat flow and on the surface friction than in the case of impermeable surface.

NOTATION

u and v	are the lengthwise and cross stream velocity components;
ρ and p	are the gas density and pressure;
C_p	is the gas specific heat capacity at constant pressure;
γ	is the adiabatic index;
Re and Pr	are the Reynolds and Prandtl numbers;
ν	is the kinematic viscosity coefficient;
σ_T and σ_E	are the accommodation coefficients of tangential impulse and of energy.

Subscripts

w and 0 refer to parameters on the cylinder surface and at a critical point of external flow respectively.

LITERATURE CITED

1. V. G. Leitsina and N. V. Pavlyukevich, Inzh.-Fiz. Zh., 12, No. 3 (1967).
2. H. Schlichting, Theory of Boundary Layer [Russian translation], Nauka, Moscow (1969).
3. D. R. Chapman and M. V. Rubezin, in: Mechanics [Russian translation], IL, No. 4 (1950).
4. V. P. Shidlovskii, Introduction to Dynamics of Rarefied Gases [in Russian], Nauka, Moscow (1965).
5. H. Schlichting and K. Bussman, Schriften der deutschen Akademie d. Luftfahrtforschung, 7B, No. 2 (1943).